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# A Principle of Randomization for Coincidence Points with Applications

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**Abstract**—In this paper, a randomizing theorem of coincidence points for set-valued mappings is shown. As applications, we utilize this result to obtain some new random coincidence points theorems. © 1998 Elsevier Science Ltd. All rights reserved.

**Keywords**—Random coincidence points theorem, Set-valued mappings, Polish space, Measurable selection theorem.

## 1. INTRODUCTION AND PRELIMINARIES

Random fixed-point theory has received much attention in recent years (see, e.g., [1–24], etc.). In this paper, we establish a principle of randomization of coincidence points for set-valued mappings with a stochastic domain. As applications, we utilize this result to obtain some new random coincidence points theorems that generalize results existing in the literature.

Let $(\Omega, \Sigma, \mu)$ be a complete $\sigma$-finite measure space and $X$ a Polish space (that is, a complete, separable, metrizable space). We will denote by $d(\cdot, \cdot)$, a metric compatible with the topology of $X$. Let $\mathcal{B}(X)$ be the $\sigma$-algebra of Borel subsets of $X$, $P_f(X)$ the family of all nonempty closed subsets of $X$, $P_{cc}(X)$ the family of all nonempty weakly compact convex subsets of $X$, $N$ the set of all positive integers, and $R_+ = [0, \infty)$.

Recall (see [25,26]) that a set-valued mapping $F : \Omega \to 2^X \backslash \{\emptyset\}$ is said to be measurable if either of the following two equivalent statements hold:

(1) for all $U \subset X$ open, $\{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\} \in \Sigma$,
(2) for all $y \in X$, $\omega \to d(y, F(\omega)) = \inf\{d(y,x) : x \in F(\omega)\}$ is measurable. If $F(\cdot)$ is closed valued, then (1) and (2) above are equivalent to
(3) $F$ has graph measurability, i.e., $Gr(F) = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times \mathcal{B}(X)$.

Following Engl [5] and Schal [27], we will say that a measurable $F(\cdot)$ is separable if there exists a countable set $D \subset X$ such that $F(\omega) = \overline{F(\omega) \cap D}$, for all $\omega \in \Omega$. Also a set-valued mapping $T : Gr(F) \to 2^X \backslash \{\emptyset\}$ is said to be an "adjective" random mapping with stochastic domain $F(\cdot)$, if for all $x \in X$ and all $U \subset X$ open, $\{\omega \in \Omega : T(\omega, x) \cap U \neq \emptyset,\ x \in F(\cdot)\} \in \Sigma$ and for every $\omega \in \Omega, x \to T(\omega, x)$ is "adjective" on $F(\omega)$. Random operators with stochastic domain were first introduced and studied by Engl [7] (single-valued case) and [5] (set-valued case).

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If $Y$ is a Hausdorff topological space, $(Z,d)$ is a metric space, and $G : Y \to 2^Z\backslash\{\emptyset\}$ is closed valued, we say that $G(\cdot)$ is $H$-continuous, if it is continuous from $Y$ into the closed subsets of $Z$ with the Hausdorff (generalized) metric $H(\cdot,\cdot)$. Also, we say that $G : Y \to 2^Z\backslash\{\emptyset\}$ is $d$-continuous, if $y \to d(z, G(y))$ is continuous for all $z \in Z$. Note that an $H$-continuous set-valued mapping is $d$-continuous. Finally, we say that mapping $\phi : Y \to (-\infty, +\infty)$ is lower semicontinuous (l.s.c.) if for all $y_0 \in Y$, $\liminf_{y\to y_0} \phi(y) \geq \phi(y_0)$.

LEMMA 1.1. *(See [26].) Let $Y$ be a separable metric space and $Z$ a metric space. Suppose that $f : \Omega \times Y \to Z$ satisfies the following conditions:*

(i) *for all $x \in Y$, $f(\cdot, x) : \Omega \to Z$ is measurable;*
(ii) *for all $\omega \in \Omega$, $f(\omega, \cdot) : Y \to Z$ is continuous.*

*If $g : \Omega \to Y$ is measurable, then $f(\cdot, g(\cdot)) : \Omega \to Z$ is measurable.*

LEMMA 1.2. *(See [16].) Let $X$ and $Y$ be Polish spaces. If $F : \Omega \to P_f(X)$ is a separable measurable set-valued mapping, and $T : Gr(F) \to Y$ is a continuous random mapping with stochastic domain $F(\cdot)$, then there exists $\hat{T} : \Omega \times X \to Y$ a Caratheodory mapping such that $\hat{T}|_{Gr(F)} = T$.*

Let $(X,d)$ be a metric space, and $\phi : X \to R$ a function. We define a partial order $\leq$ on $X$ as follows:

$$x \leq y \Longleftrightarrow d(x,y) \leq \phi(x) - \phi(y), \qquad \forall\, x, y \in X.$$

Let $X_\phi = (X, \leq)$.

LEMMA 1.3. *(See [3].) Let $(X,d)$ be a complete metric space and $\phi : X \to R$ a lower semicontinuous function. If the function $\phi$ is bounded from below, then $X_\phi$ has a maximal element.*

## 2. A PRINCIPLE OF RANDOMIZATION OF COINCIDENCE POINTS FOR SET-VALUED MAPPINGS

In this section, we establish a principle of randomization of coincidence points for set-valued mappings, which offers a useful tool for obtaining the stochastic analogue of the deterministic coincidence points theorems.

THEOREM 2.1. *Let $X$ be a Polish space, $F : \Omega \to P_f(X)$ a separable measurable set-valued mapping, $g : Gr(F) \to X$ a continuous random mapping with stochastic domain $F(\cdot)$, and $T_i : Gr(F) \to P_f(X)$ $d$-continuous random mappings with stochastic domain $F(\cdot)$, $i = 1, 2, \dots$. Suppose that for any $\omega \in \Omega$, there exists $y \in F(\omega)$ such that $g(\omega, y) \in T_i(\omega, y)$, $i = 1, 2, \dots$. Then there exists a measurable mapping $x : \Omega \to X$ such that, for all $\omega \in \Omega$,*

$$x(\omega) \in F(\omega), \qquad g(\omega, x(\omega)) \in T_i(\omega, x(\omega)), \qquad i = 1, 2, \dots.$$

PROOF. Let $G(\omega) = F(\omega) \times X$. Since $F(\cdot)$ is separable measurable, so is $G(\cdot)$. For any $i \in N$, we define $\varphi_i : Gr(G) \to R_+$ as the following:

$$\varphi_i(\omega, x, y) = d(y, T_i(\omega, x)), \qquad i = 1, 2, \dots.$$

Then for any $(x,y) \in X \times X$, for any $\lambda > 0$, and for any $i \in N$, we have

$$\begin{aligned}&\{\omega \in \Omega : \varphi_i(\omega, x, y) < \lambda, (x,y) \in G(\omega)\}\\ &\qquad = \{\omega \in \Omega : T_i(\omega, x) \cap [z \in X : d(z,y) < \lambda] \neq \emptyset, (x,y) \in G(\omega)\} \in \Sigma.\end{aligned}$$

Also note that for fixed $\omega \in \Omega$, if $(x_n, y_n) \to (x,y)$, then for any $i \in N$, it follows from the $d$-continuity of $T_i$ that

$$\begin{aligned}|d(y_n, T_i(\omega, x_n)) - d(y, T_i(\omega, x))| &\leq |d(y_n, T_i(\omega, x_n)) - d(y, T_i(\omega, x_n))|\\ &\quad + |d(y, T_i(\omega, x_n)) - d(y, T_i(\omega, x))|\\ &\leq d(y_n, y) + |d(y, T_i(\omega, x_n)) - d(y, T_i(\omega, x))| \to 0.\end{aligned}$$

So $(x, y) \to \varphi_i(\omega, x, y)$ is continuous, for all $i \in N$. Hence, $\varphi_i(\cdot,\cdot,\cdot)$ is a continuous random mapping with the separable, measurable stochastic domain $G(\cdot)$, $i = 1, 2, \ldots$. By Lemma 1.2, there exists $\hat{\varphi}_i : \Omega \times X \times X \to (-\infty, +\infty)$ a Caratheodory extension of $\varphi_i(\cdot,\cdot,\cdot)$, $i = 1, 2, \ldots$.

Since $g : Gr(F) \to X$ is a continuous random mapping with the separable, measurable stochastic domain $F(\cdot)$, it follows from Lemma 1.2 that there exists $\hat{g} : \Omega \times X \to X$ a Caratheodory extension of $g$ such that $\hat{g}|_{Gr(F)} = g$.

For fixed $(x, y) \in X \times X$, we know that $\hat{g}(\cdot, y) : \Omega \to X$ and $\hat{\varphi}_i(\cdot, x, y) : \Omega \to (-\infty, +\infty)$ are measurable, $i = 1, 2, \ldots$. Also, for fixed $\omega \in \Omega$, the mapping $\hat{\varphi}_i(\omega,\cdot,\cdot) : X \times X \to (-\infty, +\infty)$ is continuous, $i = 1, 2, \ldots$. Hence, for fixed $(x, y) \in X \times X$, it follows from Lemma 1.1 that $\omega \to \hat{\varphi}_i(\omega, x, \hat{g}(\omega, y))$ is measurable, $i = 1, 2, \ldots$. Since $y \to \hat{g}(\omega, y)$ is continuous, we know that $(x, y) \to \hat{\varphi}_i(\omega, x, \hat{g}(\omega, y))$ is continuous.

For any $i \in N$, we define set-valued mappings $\hat{T}_i : \Omega \times X \to P_f(X)$ as the following:

$$\hat{T}_i(\omega, x) = \begin{cases} T_i(\omega, x), & (\omega, x) \in Gr(F), \\ C, & (\omega, x) \notin Gr(F), \end{cases}$$

where $C \in P_f(X)$ is arbitrary. Now we consider the set-valued mapping $K : \Omega \to 2^X$

$$K(\omega) = \left\{ x \in F(\omega) : \hat{g}(\omega, x) \in \bigcap_{i \in N} \hat{T}_i(\omega, x) \right\}.$$

By the assumption, $K(\omega) \neq \emptyset$, for all $\omega \in \Omega$. Since

$$\begin{aligned} Gr(K) &= \left\{ (\omega, x) \in Gr(F) : \hat{g}(\omega, x) \in \bigcap_{i \in N} \hat{T}_i(\omega, x) \right\} \\ &= \bigcap_{i \in N} \left\{ (\omega, x) \in Gr(F) : \hat{g}(\omega, x) \in \hat{T}_i(\omega, x) \right\} \\ &= \bigcap_{i \in N} \{ (\omega, x) \in Gr(F) : \hat{\varphi}_i(\omega, x, \hat{g}(\omega, x)) = 0 \} \in \Sigma \times \mathcal{B}(X), \end{aligned}$$

by Aumann's selection theorem (see [28]), there exists a measurable selection $x : \Omega \to X$, such that for all $\omega \in \Omega$, $x(\omega) \in K(\omega)$, i.e., for all $\omega \in \Omega$,

$$x(\omega) \in F(\omega), \qquad g(\omega, x(\omega)) \in T_i(\omega, x(\omega)), \qquad i = 1, 2, \ldots.$$

This completes the proof.

REMARK. Theorem 2.1 extends some known results such as Theorem 3.1 of [16] and Theorem 1 of [15].

## 3. APPLICATIONS

In this section, we use Theorem 2.1 to obtain some new random coincidence points theorems.

THEOREM 3.1. *Let $X$ be a Banach space, $F : \Omega \to P_{cc}(X)$. Let $g : Gr(F) \to X$ be a mapping and $T : Gr(F) \to P_f(X)$ be a mapping such that the mapping $x \to d(g(\omega, x), T(\omega, x))$ is l.s.c. for all $\omega \in \Omega$. If for all $\omega \in \Omega$,*

$$\inf\{d(g(\omega, x), T(\omega, x)) : x \in F(\omega)\} = 0 \tag{3.1}$$

*and for all $\omega \in \Omega$, $x, y \in F(\omega)$, $0 \le \lambda \le 1$, $u = \lambda x + (1 - \lambda) y$, we have*

$$d(g(\omega, u), T(\omega, u)) \le \Phi(\max[d(g(\omega, x), T(\omega, x)), d(g(\omega, y), T(\omega, y))]), \tag{3.2}$$

*where $\Phi : R_+ \to R_+$ is nondecreasing, continuous from the right at 0 with $\Phi(0) = 0$. Then*

(i) *for all $\omega \in \Omega$, there exists $x \in F(\omega)$, such that $g(\omega, x) \in T(\omega, x)$;*

(ii) *if $F : \Omega \to P_{cc}(X)$ is a separable measurable set-valued mapping, $g : Gr(F) \to X$ a continuous random mapping with stochastic domain $F(\cdot)$, and $T : Gr(F) \to P_f(X)$ a $d$-continuous random mapping with stochastic domain $F(\cdot)$, then there exists a measurable mapping $x : \Omega \to X$ such that for all $\omega \in \Omega$,*

$$x(\omega) \in F(\omega), \qquad g(\omega, x(\omega)) \in T(\omega, x(\omega)).$$

PROOF. (i) Take a decreasing sequence $\{c_n\}$ of positive numbers such that $c_n \downarrow 0$ and $\Phi(c_{n+1}) < c_n$, for all $n \geq 1$. For any $\omega \in \Omega$ and $n = 1, 2, \dots$, let

$$A_n(\omega) = \{x \in F(\omega) : d(g(\omega, x), T(\omega, x)) \leq c_n\}.$$

By (3.1) and $x \to d(g(\omega, x), T(\omega, x))$ is l.s.c., we know that $A_n(\omega) \neq \emptyset$. For all $n \geq 1$ and $\omega \in \Omega$, we let $x_n(\omega) \in A_n(\omega)$ and

$$K_n(\omega) = \mathrm{co}\{x_k(\omega) : k \geq n + 1\},$$

where $\mathrm{co}(A)$ denote the convex hull of $A$.

We shall first claim that for $n \leq m$,

$$K_{n,m}(\omega) = \mathrm{co}\{x_k(\omega) : n + 1 \leq k \leq m + 1\}$$

satisfies that

$$K_{n,m}(\omega) \subset A_n(\omega), \qquad \forall n \geq 1. \tag{3.3}$$

It is clear that (3.3) is valid for $n = m$. Now, we suppose that (2.3) is valid for $n = s \leq m$, then for $x \in K_{s-1,m}$, we have $x = \lambda x_s + (1 - \lambda)y$ for some $0 \leq \lambda \leq 1$ and $y \in K_{s,m} \subset A_s$. Using (3.2), we have

$$d(g(\omega, x), T(\omega, x)) \leq \Phi(\max[d(g(\omega, x_s), T(\omega, x_s)), d(g(\omega, y), T(\omega, y))]) \leq \Phi(c_s) < c_{s-1}.$$

Hence, $K_{n,m}(\omega) \subset A_n(\omega)$, for all $\omega \in \Omega$ , $m \geq n \geq 1$ and then $K_n(\omega) \subset A_n(\omega)$. Thus, $\overline{K_n(\omega)} \subset A_n(\omega)$. Since $\overline{K_{n+1}(\omega)} \subset \overline{K_n(\omega)}$ and $\overline{K_n(\omega)}$ is weakly compact, we have

$$\bigcap_{n=1}^{\infty} A_n(\omega) \supset \bigcap_{n=1}^{\infty} \overline{K_n(\omega)} \neq \emptyset.$$

Take $x \in \cap_{n=1}^{\infty} A_n(\omega)$, then $x \in F(\omega)$, such that $g(\omega, x) \in T(\omega, x)$.

(ii) Since $F : \Omega \to P_{cc}(X)$ is a separable measurable set-valued mapping, we know that $F$ is a closed valued mapping and so $F : \Omega \to P_f(X)$ is a separable measurable set-valued mapping. Moreover, since $g : Gr(F) \to X$ is a continuous random mapping with stochastic domain $F(\cdot)$ and $T : Gr(F) \to P_f(X)$ a $d$-continuous random mapping with stochastic domain $F(\cdot)$, it follows from (i) and Theorem 2.1 that there exists a measurable mapping $x : \Omega \to X$ such that for all $\omega \in \Omega$,

$$x(\omega) \in F(\omega), \qquad g(\omega, x(\omega)) \in T(\omega, x(\omega)).$$

This completes the proof.

REMARK. Theorem 3.1 extends Theorem 1 of [29].

THEOREM 3.2. *Let $X$ be a Banach space, $F : \Omega \to P_{cc}(X)$ a separable measurable set-valued mapping, $g : Gr(F) \to X$ a continuous random mapping with stochastic domain $F(\cdot)$, and $T : Gr(F) \to P_f(X)$ an $H$-continuous random mapping with stochastic domain $F(\cdot)$ such that conditions (3.1) and (3.2) hold, then there exists a measurable mapping $x : \Omega \to X$ such that for all $\omega \in \Omega$,*

$$x(\omega) \in F(\omega), \qquad g(\omega, x(\omega)) \in T(\omega, x(\omega)).$$

PROOF. Since $T$ is $H$-continuous, we know that $T$ is $d$-continuous. For any $\omega \in \Omega$ and any $x, y \in F(\omega)$, we have

$$d(g(\omega,x), T(\omega,x)) \le d(g(\omega,x), g(\omega,y)) + d(g(\omega,y), z) + d(z, T(\omega,x)), \qquad \forall\, z \in T(\omega,y).$$

Hence,

$$d(g(\omega,x), T(\omega,x)) \le d(g(\omega,x), g(\omega,y)) + d(g(\omega,y), z) + H(T(\omega,y), T(\omega,x)), \qquad \forall\, z \in T(\omega,y)$$

and so

$$d(g(\omega,x), T(\omega,x)) \le d(g(\omega,x), g(\omega,y)) + d(g(\omega,y), T(\omega,y)) + H(T(\omega,y), T(\omega,x)).$$

Similarly, we can prove the following is true:

$$d(g(\omega,y), T(\omega,y)) \le d(g(\omega,x), g(\omega,y)) + d(g(\omega,x), T(\omega,x)) + H(T(\omega,y), T(\omega,x)).$$

Combining the above inequalities, we have

$$|d(g(\omega,x), T(\omega,x)) - d(g(\omega,y), T(\omega,y))| \le d(g(\omega,x), g(\omega,y)) + H(T(\omega,x), T(\omega,y)).$$

By the continuity of $g$ and the $H$-continuity of $T$, it is easy to see that the mapping $x \to d(g(\omega,x), T(\omega,x))$ is continuous, hence it is l.s.c., for all $\omega \in \Omega$. By Theorem 3.1, there exists a measurable mapping $x : \Omega \to X$ such that for all $\omega \in \Omega$,

$$x(\omega) \in F(\omega), \qquad g(\omega, x(\omega)) \in T(\omega, x(\omega)).$$

This completes the proof.

THEOREM 3.3. *Let $X$ be a Polish space, $F : \Omega \to P_f(X)$ a separable measurable set-valued mapping, $g : Gr(F) \to X$ a continuous random mapping with stochastic domain $F(\cdot)$, and $T_i : Gr(F) \to P_f(X)$ $d$-continuous random mappings with stochastic domain $F(\cdot)$, $i = 1, 2, \ldots$. Suppose that for any $\omega \in \Omega$, $g(\omega, F(\omega)) = X$. If for any $x \in F(\omega)$ and $i \in N$, when $g(\omega, x) \notin \cap_{i \in N} T_i(\omega, x)$, there exists a $y_i \in T_i(\omega, x) \backslash \{g(\omega, x)\}$ such that*

$$d(g(\omega,x), y_i) \le \phi(\omega, g(\omega,x)) - \phi(\omega, y_i),$$

*where $\phi : Gr(F) \to R_+$ such that for all $\omega \in \Omega$, $\phi(\omega, \cdot) : F(\omega) \to R_+$ is l.s.c., then there exists a measurable mapping $x : \Omega \to X$ such that for all $\omega \in \Omega$,*

$$x(\omega) \in F(\omega), \qquad g(\omega, x(\omega)) \in T_i(\omega, x(\omega)), \quad i = 1, 2, \ldots.$$

PROOF. For any given $\omega \in \Omega$, and $x', y' \in X$, since $g(\omega, F(\omega)) = X$, there exist $x, y \in F(\omega)$ such that $x' = g(\omega, x)$ and $y' = g(\omega, y)$. We define a partial order $\le$ on $X$ as follows:

$$x' \le y' \iff d(g(\omega,x), g(\omega,y)) \le \phi(g(\omega,x)) - \phi(g(\omega,y)).$$

By Lemma 1.3, $X_\phi = (X, \leq)$ has a maximal element $x_0' \in X$. Since $g(\omega, F(\omega)) = X$, there exists $x_0 \in F(\omega)$ such that $x_0' = g(\omega, x_0)$.

Furthermore, for $x_0 \in F(\omega)$, by the hypothesis, for each $i \in N$, if $g(\omega, x_0) \notin \cap_{i \in N} T_i(\omega, x_0)$, then there exists a $y_i \in T_i(\omega, x_0) \backslash \{g(\omega, x_0)\}$ such that

$$d(g(\omega, x_0), y_i) \leq \phi(\omega, g(\omega, x_0)) - \phi(\omega, y_i). \tag{3.4}$$

Also, since $g(\omega, F(\omega)) = X$, there exists $x_i \in F(\omega)$ such that $y_i = g(\omega, x_i)$. It follows from (3.4) that $g(\omega, x_0) \leq y_i$. Since $x_0' = g(\omega, x_0)$ is the maximal element, we have

$$g(\omega, x_0) = y_i = g(\omega, x_i) \in T_i(\omega, x_0) \backslash \{g(\omega, x_0)\}.$$

This is a contradiction. Hence, $g(\omega, x_0) \in \cap_{i \in N} T_i(\omega, x_0)$. By Theorem 2.1, we know that there exists a measurable mapping $x : \Omega \to X$ such that for all $\omega \in \Omega$,

$$x(\omega) \in F(\omega), \qquad g(\omega, x(\omega)) \in T_i(\omega, x(\omega)), \quad i = 1, 2, \ldots.$$

This completes the proof.

REMARK. Theorem 3.3 is a random generalization of the set-valued Caristi's fixed-point theorem.

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